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Equilibrium Points of NonAtomic Games:
Asymptotic Results

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Abstract

An elementary proof, as well as an asymptotic version of David Schmeidler's theorem on the Equilibrium Points of Nonatomic Games is presented.

Equilibrium Points of NonAtomic Games: Asymptotic Results

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The theory of n-person non zero-sum games has possessed an equilibrium theorem for many years but there are still not many results characterising the solutions of such games. In 1973 David Schmeidler (1) partially remedied this situation for a class of games defined on a nonatomic measure space by showing that there existed equilibria for such games in which almost every individual uses a pure strategy; Schmeidler's paper does not establish any asymptotic results and he uses fairly sophisticated properties of the integrals of set-valued functions to establish his results. In this paper we provide an asymptotic version of Schmeidler's result using only elementary arguments.

We begin with a finite set of players, 1, 2, . . ., n, and the non-negative unit simplex s_m on m-dimensional commodity space. An nm dimensional vector is announced, allocating a vector in the simplex to each of the n players. Every player disregards the bundle allocated to him, but keeping in mind the vectors allocated to all other players, chooses a bundle in the simplex that maximises his utility. An individual's utility is dependent both upon what he gets as well as what every other player gets; the utility function is linear in the bundle chosen by the player and continuous as a function of the vectors allocated to all other players. If $\hat{x}_t$ denotes the vector of allocations across all players but player t, then the t^{th} player's problem can be stated as

$$\max_{x \in S_m} x \cdot V(t,\hat{x}_t) = x_{t1}V_{t1}(\hat{x}_t) + \ldots + x_{tm} V_{tm}(\hat{x}_t)$$

where each V_{tj} is a continuous function of $\hat{x}_t$. Call this problem A. A choice of strategies $(x_1, \ldots, x_n)$ is a (Nash) equilibrium if, for all players and all $p \varepsilon S_m$, $x_t \cdot V(t,\hat{x}_t) \geq p \cdot V(t,\hat{x}_t)$. The existence of equilibria for problem A is a special case of classical results and will be assumed. In general, the equilibrium vectors chosen will not be pure strategies--i.e., vectors of the form (1, 0, . . ., 0), (0, 1, . . ., 0), etc. Let us now specialise the $V_{jt}(\hat{x}_t)$ so that the V_{jt} depend only on the average sum

$$\frac{\sum_{j \neq t} x_j}{n} .$$

Call this problem B. Schmeidler's principal theorem shows that, for games played on nonatomic measure spaces, there exists an equilibrium for problem B in which almost every player--i.e., except for a set of measure zero--uses a pure strategy. This result may alternatively be expressed as follows:

The solution to problem A, with n players, may be written as an n x m matrix, each row of which is a vector in the unit simplex of m-space. Under the additional hypotheses of problem B, Schmeidler showed that there exist solutions in which the optimal strategy for "most" traders consists of a unit vector of the form (0, 1, 0, . . ., 0). The asymptotic version to be proved here will show that when problem B is played by n agents, then there exist Nash "equilibria" in which all but n - m agents will be approximately maximal when using a pure strategy; the distance from full maximality for these n - m agents tends to zero as $n \to \infty$, while the number of agents who are not even approximately maximal while using pure strategies is bounded above by m, the fixed dimension of commodity space.

The proof will depend heavily upon the two following observations.

1. Denote problem B' as the variant of problem B in which each V_{tk} depends not on $\sum_{j \neq t} x_j/n$, but on $\sum_{j=1}^{n} x_j/n$. As $\sum_{j \neq t} x_j/n$ differs from $\sum_{j=1}^{n} x_j/n$ by at most $(\pm 1/n, \ldots, \pm 1/n)$, the continuity of the V_{tj} ensure that the difference between the optimal solutions to problems B and B' will tend to zero as n tends to ∞. For our asymptotic result, it will therefore suffice if we solve problem B' exactly.

2. As we are maximising a linear function over a polyhedral convex set, if a maximum is reached at a convex combination of vertices, it is also reached at every one of these vertices--e.g., if the optimal vector is $(x_1, x_2, 0, \ldots, 0)$, then the optimal value is also reached at either of $(1, 0, \ldots, 0)$ or $(0, 1, 0, \ldots, 0)$.

The Nash Equilibrium allocation that is guaranteed us by the general results on n person games may be represented in matrix form as below, with the rows adding up to one, and the columns adding up to $K_1, \ldots, K_m$ respectively.

$$\begin{array}{ccc|c} x_{11} & \cdots & x_{1m} & 1 \\ \vdots & & & \\ x_{n1} & & x_{nm} & 1 \\ \hline K_1 & & K_m & n \\ \hline \end{array}$$

Under the additional assumptions of problem B', can we rearrange the matrix such that, for all but a fixed set of traders, each row is a unit basis vector, with the 1 in a position which had a positive entry?

Were it not for the condition that we must not introduce a positive entry in an element which is zero, the desired result would follow directly from the Shapley-Folkman Theorem[1] upon noting that the simplex is nothing but the convex hull of the unit vectors in m-space.

In order to make the Shapley-Folkman Theorem applicable, all we have to do is to define a new simplex for each row. Thus, if in the first row only, the first, fourth, and fifth entries are positive, let S_1 denote the vectors (1, 0, 0, 0, . . ., 0), (0, 0, 0, 1, 0, . . ., 0), and (0, 0, 0, 0, 1, 0, . . ., 0). If the second row has only the second and third entries positive, let S_2 denote the vectors (0, 1, 0 . . ., 0) and (0, 0, 1, 0, . . ., 0). Do this for each row and define an appropriate S_j for the j^{th} row.

$(K_1, \ldots, K_m)$ lies in the convex hull of ΣS_j. Hence, by the Shapley-Folkman Theorem there exists a way of choosing vectors from con S_j such that

$$(K_1, \ldots, K_m) = \sum_{J_1} S_j + \sum_{J_2} \text{con } S_j,$$

where the indices J_1 and J_2 are such that $J_1 + J_2 = n$ and $|J_2| \leq m$. Hence, except for at most m players, other individuals are at an optimum using a pure strategy. This proves the theorem. More formally:

Let G_n denote an n-person non-cooperative game in which every agent has the m-dimensional Euclidean simplex as its strategy space. The pay-off to each individual is linear in his own strategy and depends on the sum of everyone else's activities. Thus,

$$U_1(x_1; x_1, \ldots, x_n) = x_{11}V_{11}(\sum_{j=2}^{n} x_j) + \ldots + x_{1m}V_{1m}(\sum_{j=2}^{n} x_j).$$

Let $U_1^*, \ldots, U_n^*$ denote the utilities obtained at the Nash equilibrium point.

<u>Theorem</u>: For any $\varepsilon > 0$, there exists n such that G_n possesses an approximate Nash solution with utilities $U_1^{**}, \ldots, U_n^{**}$ with the following property: $|U_j^* - U_j^{**}| < \varepsilon$ for at least n - m agents.

Note

[1] The Shapley-Folkman Theorem (see Cassels, 1975) states: If S_j, j=1, . . ., n, is a collection of sets in m-dimensional space, n > m, and con S_j is the convex hull of S_j, then for any $x \varepsilon \sum_{j=1}$ con s_j, there exists a representation of x of the following form:

$$x = \sum_{J_1} y_j + \sum_{J_2} z_j,$$

where $y_j \varepsilon S_j$, $z_j \varepsilon$ con S_j, $J_1 + J_2 = n$, and the number of elements in J_2 is no greater than m.

References

1. D. Schmeidler, "Equilibrium Points of NonAtomic Games," Journal of Statistical Physics (April 1973), 295-300.

2. J. W. S. Cassels, "Measures of the non-convexity of sets and the Shapley-Folkman-Starr Theorem," Math. Proc. Camb. Phil. Soc. (1975) 78, 433-436.